Variational Quadratic Shape Functions for Polygons and Polyhedra Supplemental Material 1

ASTRID BUNGE, TU Dortmund University, Germany
PHILIPP HERHOLZ, ETH Zurich, Switzerland
OLGA SORKINE-HORNUNG, ETH Zurich, Switzerland
MARIO BOTSCH, TU Dortmund University, Germany
MICHAEL KAZHDAN, Johns Hopkins University, USA

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1 LAGRANGE ELEMENT REPRODUCTION

Given a simplex σ in 2D or 3D, let $\{\phi_i^*\}$ be the standard quadratic Lagrange elements, let $\{\psi_i\}$ be Lagrange elements on the virtually refined simplex, and let $\{\phi_i\}$ be the linear combinations of $\{\psi_i\}$ that satisfy the Lagrange interpolation property and minimize the gradient discontinuity energy.

Claim 1. *The two sets of functions are identical* $\{\phi_i\} = \{\phi_i^*\}$.

To prove the claim we formulate several lemmas.

Lemma 1.1. *Given polynomials* P_1 *and* P_2 *of degree* d *in* $\mathbb{R}^D$ *whose values and partial derivatives up to order* $d-1$ *agree on the hyperplane* $x_1 = 0$, *there exists* $\alpha \in \mathbb{R}$ *such that:*

$$P_1(x_1, \ldots, x_D) = P_2(x_1, \ldots, x_D) + \alpha \cdot x_1^d.$$

Proof. Expanding the polynomials as:

$$P_i(x_1, \ldots, x_D) = \sum_{j_1 + \ldots + j_D \le d} a^i_{j_1 \ldots j_D} \cdot x_1^{j_1} \cdots x_D^{j_D}$$

we note that the coefficients $a^i_{j_1 \ldots j_D}$ must agree whenever $j_1 < d$. To see this, take the j_1-th partial derivative with respect to x_1 and consider the resulting polynomials Q_i in $D-1$ variables:

$$\begin{aligned} Q_i(x_2, \ldots, x_D) &= \frac{\partial^{j_1}}{\partial x_1^{j_1}} P_i(0, x_2, \ldots, x_D) \\ &= \sum_{j_2 + \ldots + j_D \le d - j_1} (j_1!) \cdot a^i_{j_1 \ldots j_D} \cdot x_2^{j_2} \cdots x_D^{j_D}. \end{aligned}$$

Since we take $j_1 < d$ derivatives, the polynomials Q_1 and Q_2 are equal, implying that their coefficients $a^i_{j_1 \ldots j_D}$ are the same. Thus, the polynomials P_1 and P_2 can only differ in their x_1^d term. ◻

Corollary 1.1.1. *Although we required the values and derivatives (up to order* $d-1$*) of* P_1 *and* P_2 *to agree on the hyperplane* $x_1 = 0$, *Lemma 1 holds if they agree on any open (non-empty) subset of the hyperplane. (This follows from the fact that polynomials are analytic.)*

Corollary 1.1.2. *We can replace the condition that the polynomials* P_1 *and* P_2 *agree on the hyperplane* $x_1 = 0$ *with the condition that* P_1 *and* P_2 *agree on a hyperplane defined by the linear system* $\langle \mathbf{x}, \mathbf{v} \rangle = 0$ *for any* $\mathbf{v} \in \mathbb{R}^D$. *In this case there exists* $\alpha \in \mathbb{R}$ *such that:*

$$P_1(\mathbf{x}) = P_2(\mathbf{x}) + \alpha \cdot \langle \mathbf{x}, \mathbf{v} \rangle^d.$$

Lemma 1.2. *Given a vector* $\mathbf{v}$, *denote by* $\Lambda_{\mathbf{v}} : \mathbb{R}^D \to \mathbb{R}$ *the linear function* $\Lambda_{\mathbf{v}}(\mathbf{x}) \equiv \langle \mathbf{v}, \mathbf{x} \rangle$. *Given* $D+1$ *vectors* $\{\mathbf{v}_0, \ldots, \mathbf{v}_D\}$ *in* $\mathbb{R}^D$, *no* D *of which are linearly dependent, the set of functions:*

$$\left\{ \Lambda^d_{\mathbf{v}_i}(\mathbf{x}) = \langle \mathbf{v}_i, \mathbf{x} \rangle^d \right\}$$

are linearly independent for all $d > 1$.

Proof. The statement is invariant under linear transformation so we can assume, without loss of generality, that $\{\mathbf{v}_1, \ldots, \mathbf{v}_D\}$ are the coordinate axes. By linear independence, we have: $\mathbf{v}_0 = (v_1, \ldots, v_D)$ where *all* of the v_i are non-zero. In this coordinate system we have:

$$\Lambda_{\mathbf{v}_i}(x_1, \ldots, x_D) = x_i^d, \quad \forall 1 \le i \le D$$

$$\Lambda_{\mathbf{v}_0}(x_1, \ldots, x_D) = \left(\sum_{i=1}^{D} v_i x_i \right)^d.$$

The $\Lambda_{\mathbf{v}_i}$ (w/ $1 \le i \le D$) are linearly independent as they are functions of different variables. In addition $\Lambda_{\mathbf{v}_0}$ cannot be expressed as the linear combination of the $\Lambda_{\mathbf{v}_i}$ (w/ $1 \le i \le D$) since $\Lambda_{\mathbf{v}_0}$ contains mixed terms while the monomials $\Lambda_{\mathbf{v}_i}$ (w/ $1 \le i \le D$) do not. ◻

We can now prove the claim. We proceed in two steps. First we show that the Lagrange basis function, $\{\phi_i^*\}$, are in the span of $\{\psi_i\}$ and that they have zero energy. Then we show that these are the only functions that have zero energy and satisfy the Lagrange interpolation property, implying that they are the unique minimizers. Proving the first statement is straight-forward. Proving the second amounts to associating functions with edges in the adjacency graph of the simplicial refinement and showing that the functions associated with cycles in the graph are linearly independent.

Authors' addresses: Astrid Bunge, TU Dortmund University, Dortmund, Germany, astrid.bunge@tu-dortmund.de; Philipp Herholz, ETH Zurich, Zurich, Switzerland, ph.herholz@gmail.com; Olga Sorkine-Hornung, ETH Zurich, Zurich, Switzerland, sorkine@inf.ethz.ch; Mario Botsch, TU Dortmund University, Dortmund, Germany, mario.botsch@tu-dortmund.de; Michael Kazhdan, Johns Hopkins University, Baltimore, MD, USA, misha@cs.jhu.edu.

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Proof. To prove that the $\{\phi_i^*\}$ are a solution, it suffices to show that each ϕ_j^* is in the span of $\{\psi_i\}$ and that the energy of each ϕ_j^* is zero. To prove that ϕ_j^* is in the span of $\{\psi_i\}$, let $\{\mathbf{p}_k\} \subset \sigma$ be the Lagrange nodes in the refined complex Σ_σ. Setting

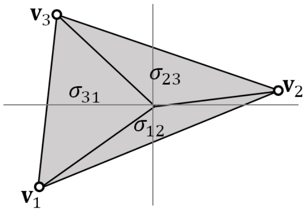
$$\phi_j(\mathbf{x}) = \sum_k \phi_j^*(\mathbf{p}_k) \cdot \psi_k(\mathbf{x})$$

we get a piecewise quadratic polynomial that agrees with the Lagrange basis functions ϕ_j^* at all refined nodes. By the Lagrange interpolation property, this implies that ϕ_j and ϕ_j^* agree on each $\sigma' \in \Sigma_\sigma$. Thus ϕ_j and ϕ_j^* agree on all of σ. Additionally, since the $\{\phi_i^*\}$ are strictly quadratic, they have no derivative discontinuity and the discontinuity energy is trivially zero.

To prove that $\{\phi_i^*\}$ is the unique minimizer we show that if the $\{\phi_i\}$ have zero energy then they must be strictly quadratic in σ. Then the proof follows from the fact that Lagrange basis functions are the only quadratic functions satisfying the interpolation property.

Since the claim is invariant to affine transformation, we assume that the simplex σ is translated so that the virtual vertex introduced in the interior of σ is at the origin, and consider two cases.

2D. We start with the case of a triangle $\sigma = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. After virtual refinement, σ is partitioned into three sub-triangles, $\Sigma_\sigma = \{\sigma_{12}, \sigma_{23}, \sigma_{31}\}$, with indices chosen so that vertex $\mathbf{v}_i$ is opposite triangle σ_{jk} for $i \neq j, k$.



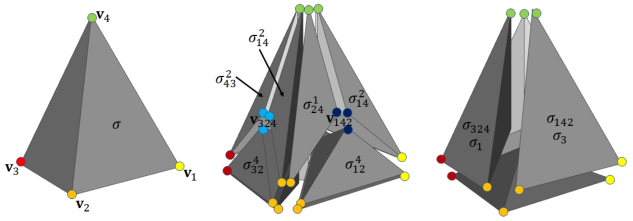
Now, given a piecewise quadratic polynomial P (strictly quadratic within each σ_{ij}) whose gradient is continuous across σ, we denote the restriction of P to σ_{ij} as P_{ij}. By Corollary 1.1.2 we have:

$$\begin{aligned} P_{12}(\mathbf{x}) &= P_{23}(\mathbf{x}) + \alpha_2 \cdot \Lambda^2_{\mathbf{v}_2^\perp}(\mathbf{x}) \\ &= P_{31}(\mathbf{x}) + \alpha_3 \cdot \Lambda^2_{\mathbf{v}_3^\perp}(\mathbf{x}) + \alpha_2 \cdot \Lambda^2_{\mathbf{v}_2^\perp}(\mathbf{x}) \\ &= P_{12}(\mathbf{x}) + \alpha_1 \cdot \Lambda^2_{\mathbf{v}_1^\perp}(\mathbf{x}) + \alpha_3 \cdot \Lambda^2_{\mathbf{v}_3^\perp}(\mathbf{x}) + \alpha_2 \cdot \Lambda^2_{\mathbf{v}_2^\perp}(\mathbf{x}) \end{aligned}$$

for real values $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. Or, equivalently:

$$0 = \alpha_1 \cdot \Lambda^2_{\mathbf{v}_1^\perp}(\mathbf{x}) + \alpha_2 \cdot \Lambda^2_{\mathbf{v}_2^\perp}(\mathbf{x}) + \alpha_3 \cdot \Lambda^2_{\mathbf{v}_3^\perp}(\mathbf{x}).$$

Since no two of the $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are collinear, by Lemma 1.2 the functions $\{\Lambda_{\mathbf{v}_i^\perp}\}$ are linearly independent so that $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and hence P is strictly quadratic in σ.



3D. Next, we consider the case of a tetrahedron $\sigma = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and assume we have a piecewise quadratic polynomial P (strictly quadratic within each $\sigma' \in \Sigma_\sigma$) whose gradient is continuous across σ. We proceed in two steps. First we show that P must be strictly quadratic within the tetrahedra defined by joining the faces of σ with the origin (i.e. the virtual vertex defined in the interior of σ). Then we show that P is strictly quadratic in σ.

Denote by $\sigma_{ijk} = \{\mathbf{0}, \mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k\}$ the tetrahedron defined by joining the face $\{\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k\}$ with the origin. Note that σ_{ijk} is not in the refined complex Σ_σ but we can express σ_{ijk} as the union of three simplices in Σ_σ. Specifically, if we denote by $\mathbf{v}_{ijk}$ the virtual vertex in the interior of the face $\{\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k\}$ and set σ_{ij}^k to be the simplex $\sigma_{ij}^k = \{\mathbf{0}, \mathbf{v}_{ijk}, \mathbf{v}_i, \mathbf{v}_j\}$ which is in the refined complex, then we have:

$$\sigma_{ijk} = \sigma_{ij}^k \cup \sigma_{jk}^i \cup \sigma_{ki}^j.$$

Again, noting that P is strictly quadratic within each each σ_{ij}^k, we denote by P_{ij}^k the restriction of P to σ_{ij}^k. Proceeding as before, cycling around edge $\{\mathbf{0}, \mathbf{v}_{ijk}\}$, we have:

$$P_{ij}^k(\mathbf{x}) = P_{ij}^k(\mathbf{x}) + \alpha_i^{jk} \cdot \Lambda^2_{\mathbf{v}_i^{jk}}(\mathbf{x}) + \alpha_k^{ij} \cdot \Lambda^2_{\mathbf{v}_k^{ij}}(\mathbf{x}) + \alpha_j^{ik} \cdot \Lambda^2_{\mathbf{v}_j^{ik}}(\mathbf{x}).$$

where $\mathbf{v}_j^{ik} = \mathbf{v}_j \times \mathbf{v}_{ijk}$ is the vector perpendicular to the face separating simplices σ_{ij}^k and σ_{jk}^i. Or, equivalently:

$$0 = \alpha_i^{jk} \cdot \Lambda^2_{\mathbf{v}_i^{jk}}(\mathbf{x}) + \alpha_j^{ik} \cdot \Lambda^2_{\mathbf{v}_j^{ik}}(\mathbf{x}) + \alpha_k^{ij} \cdot \Lambda^2_{\mathbf{v}_k^{ij}}(\mathbf{x}).$$

As before, using the linear independence of $\mathbf{v}_i^{jk}$, $\mathbf{v}_j^{ki}$, and $\mathbf{v}_k^{ij}$, and applying Lemma 1.2, it follows that $\alpha_i^{jk} = \alpha_j^{ki} = \alpha_k^{ij} = 0$ so that P is strictly quadratic in σ_{ijk}.

Finally, we show that P is strictly quadratic in σ. For simplicity, we will denote by σ_i the simplex opposite vertex $\mathbf{v}_i$:

$$\sigma_i = \{\mathbf{0}, \mathbf{v}_{i+1}, \mathbf{v}_{i+2}, \mathbf{v}_{i+3}\}$$

and we will denote by P_i the restriction of P to σ_i (which we have shown is strictly quadratic). Without loss of generality, fixing vertex $\mathbf{v}_4$, we consider simplices σ_1, σ_2, and σ_3. These meet at edge $\{\mathbf{0}, \mathbf{v}_4\}$ and, again, cycling through the simplices around the edge we get:

$$P_1(\mathbf{x}) = P_1(\mathbf{x}) + \alpha_2^4 \cdot \Lambda_{\mathbf{v}_4 \times \mathbf{v}_2}(\mathbf{x}) + \alpha_1^4 \cdot \Lambda_{\mathbf{v}_4 \times \mathbf{v}_1}(\mathbf{x}) + \alpha_3^4 \cdot \Lambda_{\mathbf{v}_4 \times \mathbf{v}_3}(\mathbf{x}).$$

Noting that $\mathbf{v}_4 \times \mathbf{v}_1$, $\mathbf{v}_4 \times \mathbf{v}_2$, and $\mathbf{v}_4 \times \mathbf{v}_3$ are linearly independent, it follows that $\alpha_1^4 = \alpha_2^4 = \alpha_3^4 = 0$ and hence P is strictly quadratic on $\sigma_1 \cup \sigma_2 \cup \sigma_3$. Repeating this argument for the $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, it follows that P is strictly quadratic on $\sigma_2 \cup \sigma_3 \cup \sigma_4$, on $\sigma_3 \cup \sigma_4 \cup \sigma_1$, and on $\sigma_4 \cup \sigma_1 \cup \sigma_2$. Thus P is strictly quadratic on all of σ. □

2 COEFFICIENT COMPUTATION

Conceptually the implementation of our method is quite simple and can be easily parallelized over the mesh elements. For simplicity, we consider a single two dimensional element e with coarse nodes C and virtual degrees of freedom $\mathcal{K}$. The extension to higher dimensions works analogously. The central task is to minimize quadratic energies of the form

$$W = \arg\min_W \sum_{i\in C}\sum_{\sigma\in\mathcal{E}^*}\int_\sigma \|\nabla^+_\sigma\phi_i - \nabla^-_\sigma\phi_i\|^2\, d\sigma \tag{1}$$

$$+\varepsilon \sum_{i\in C}\sum_{\sigma\in\Sigma(e)}\int_\sigma \|\nabla\phi_i\|^2\, d\sigma, \tag{2}$$

where $\mathcal{E}^*$ is the subset of edges in the virtual triangulation that are incident to the virtual vertex and $\Sigma(e)$ the simplicial complex obtained by refining e. We can reach the objective by expressing the energy through matrices. Focusing on a single basis function $\phi_i = w_{ii}\psi_i + \sum_{j\in\mathcal{K}} w_{ij}\psi_j$ where we introduce $w_{ii} = 1$ for convenience, we can expand the first integrand (1):

$$\begin{aligned}\int & \|\nabla^+\phi_i(\mathbf{x}) - \nabla^-\phi_i(\mathbf{x})\|^2 d\mathbf{x}\\ =\int &\sum_{j,k\in\mathcal{K}\cup\{i\}} w_{ij}w_{ik}\langle\nabla^+\psi_j(\mathbf{x}) - \nabla^-\psi_j(\mathbf{x}), \nabla^+\psi_k(\mathbf{x}) - \nabla^-\psi_k(\mathbf{x})\rangle d\mathbf{x}\\ =&\sum_{j,k\in\mathcal{K}\cup\{i\}} w_{ij}w_{ik}\underbrace{\int\langle\nabla^+\psi_j(\mathbf{x}) - \nabla^-\psi_j(\mathbf{x}), \nabla^+\psi_k(\mathbf{x}) - \nabla^-\psi_k(\mathbf{x})\rangle d\mathbf{x}}_{(\mathbf{K}_i)_{jk}}\end{aligned}$$

while the second integrand (2) becomes the Dirichlet regularizer

$$\begin{aligned}\int & \|\nabla\phi_i(\mathbf{x})\|^2 d\mathbf{x}\\ =\int &\sum_{j,k\in\mathcal{K}\cup\{i\}} w_{ij}w_{ik}\langle\nabla\psi_j(\mathbf{x}), \nabla\psi_k(\mathbf{x})\rangle d\mathbf{x}\\ =&\sum_{j,k\in\mathcal{K}\cup\{i\}} w_{ij}w_{ik}\underbrace{\int\langle\nabla\psi_j(\mathbf{x}), \nabla\psi_k(\mathbf{x})\rangle d\mathbf{x}}_{(\mathbf{S}_i)_{jk}}.\end{aligned}$$

The integrals for each pair of simplicies j, k represent the coefficients of the matrices $\mathbf{K}_i$ and $\mathbf{S}_i$ respectively and can be evaluated in closed form. The final energy can be written as

$$E(W) = \sum_i \mathbf{w}_i^\top(\mathbf{K}_i + \varepsilon\mathbf{S}_i)\mathbf{w}_i \tag{3}$$

where $\mathbf{w}_i$ stacks all weights w_{ij} with $j \in \mathcal{K}\cup\{i\}$. Concatenating all vectors $\mathbf{w}_i$ into a single vector $\mathbf{W}$ and building the block diagonal matrix

$$\mathbf{Q} = \operatorname{diag}\left(\mathbf{K}_1 + \varepsilon\mathbf{S}_1, \ldots, \mathbf{K}_{|C|} + \varepsilon\mathbf{S}_{|C|}\right),$$

the energy becomes simply $\mathbf{W}^\top\mathbf{Q}\mathbf{W}$. The remaining step is to integrate the linear constraints

1. Partition of unity:

$$\sum_{i\in C} w_{ij} = 1 \quad \text{for } j \in \mathcal{K}; \tag{4}$$

2. The virtual node positions can be expressed as affine combination of coarse nodes using the weights:

$$\mathbf{p}_j = \sum_{i\in C} w_{ij}\mathbf{p}_i \quad \text{for } j \in \mathcal{K}; \tag{5}$$

3. The weights w_{ii} are defined to be 1

into a matrix $\mathbf{E}$ and solve the quadratic optimization problem

$$\begin{aligned}\min_W &\quad \frac{1}{2}\mathbf{W}^\mathsf{T}\mathbf{Q}\mathbf{W}\\ \text{s.t.} &\quad \mathbf{E}\mathbf{W} = \mathbf{d}.\end{aligned}$$

Using the obtained weights, we can form the local prolongation matrix $\mathbf{P}$ for element e. Combining the prolongation matrices for all elements gives us the global prolongation matrix.

Variational Quadratic Shape Functions for Polygons and Polyhedra Supplemental Material 2

ASTRID BUNGE, TU Dortmund University, Germany
PHILIPP HERHOLZ, ETH Zurich, Switzerland
OLGA SORKINE-HORNUNG, ETH Zurich, Switzerland
MARIO BOTSCH, TU Dortmund University, Germany
MICHAEL KAZHDAN, Johns Hopkins University, USA

Here we prove some of the properties that we observed empirically and leveraged to develop a more efficient implementation. In particular, we will show that when an arbitrarily small Dirichlet energy is incorporated into the gradient continuity energy, the derived basis functions are uniquely defined, the partition of unity property is satisfied automatically (and does not to be enforced explicitly), and the linear precision property is satisfied automatically for faces that are planar. We start by formalizing terminology and notation (§1). We then prove that the function basis is unique, that the partition of unity property is satisfied automatically, and that the linear precision property is automatically for planar cells for both the single-level system (§2) and for the hierarchical construction (§3). We conclude with a short discussion about the generalizability of the approach (§4).

In what follows, we assume that all complexes are *pure*. That is, if the complex is D-dimensional, then every d-dimensional simplex/cell (with $d < D$) is on the boundary of some $(d+1)$-dimensional simplex/cell.

1 TERMINOLOGY AND NOTATION

We begin by presenting the terminology and notation used in the proofs. A brief description can be found in Table 1.

Simplicial Complexes

- We denote by Σ a D-dimensional simplicial complex.
- We denote by Σ_d (for $0 \leq d \leq D$) the set of d-dimensional simplices in Σ.
- Given a D-dimensional simplicial complex Σ, we denote by $\mathcal{N}(\Sigma)$ the set of quadratic Lagrange nodes of Σ.
- Given a D-dimensional simplicial complex Σ, we denote by $\mathcal{B}(\Sigma)$ the set of quadratic Lagrange basis functions on Σ. Specifically $\mathcal{B}(\Sigma) = \{\psi^d_{\eta,d}\}_{\eta\in\mathcal{N}(\Sigma)}$ where each $\psi^d_{\eta,d}$ is strictly quadratic within each simplex $\sigma \in \Sigma$ and satisfies the interpolation condition, $\psi^d_{\eta,d}(\eta') = \delta_{\eta\eta'}$ for all $\eta' \in \mathcal{N}(\Sigma)$.

Symbol	Definition
Σ	a D-dimensional simplicial complex
Σ_d	the subset of d-dimensional simplices in Σ
$\mathcal{N}(\Sigma)$	the set of Lagrange nodes of Σ
$\mathcal{B}(\Sigma)$	the set of Lagrange basis functions on Σ
C	a D-dimensional cell complex
C^d	the d-dimensional sub-complex of C
$\Sigma(C)$	the simplicial complex obtained by refining C
$\mathcal{N}^d$	the set of Lagrange nodes on $\Sigma(C^d)$
$\mathcal{B}^d$	the set of Lagrange basis functions on $\Sigma(C^d)$
$\mathring{\mathcal{B}}^d$	the subset of $\mathcal{B}^d$ indexed by interior nodes
Q_d	the quadratic energy on Span$(\mathcal{B}^d)$
$\langle\cdot,\cdot\rangle_d$	the symmetric bilinear form defined by Q_d
$\|\cdot\|_d^2$	the squared norm Q_d
$\mathcal{B}^d_{d'}$	the function basis on $\Sigma(C^d)$ indexed by $\mathcal{N}^{d'}$ $(d' \leq d)$
$\psi^d_{\eta,d'}$	the function in $\mathcal{B}^d_{d'}$ indexed by node $\eta \in \mathcal{N}^{d'}$

Table 1. Summary of notation

Linear Precision

- Given a domain embedded in Euclidean space, $\Omega \subset \mathbb{R}^n$, we say that a set of functions $\{\varphi_i\} : \Omega \to \mathbb{R}$, has linear precision if for every linear function $L : \mathbb{R}^n \to \mathbb{R}$, there exist coefficients $\{x_i\} \subset \mathbb{R}$ with:
$$L\big|_\Omega = \sum_i x_i \cdot \varphi_i.$$

Cell Complexes and Their Refinement

- We denote by C a D-dimensional cell complex.
- We denote by C^d (for $0 \leq d \leq D$) the d-dimensional subcomplex obtained by only considering cells in C with dimension less than or equal to d. In particular, $C^0 \subset \cdots \subset C^D = C$.
- Assuming that there is a virtual vertex associated with each d-dimensional cell in C (for all $d > 1$), we denote by $\Sigma(C)$ the simplicial complex obtained by refining C.
- We set $\mathcal{N}^d \equiv \mathcal{N}\big(\Sigma(C^d)\big)$ to be the set of quadratic Lagrange nodes defined on the simplicial refinement of C^d. Because the complex C is pure, we have $\mathcal{N}^0 \subset \cdots \subset \mathcal{N}^D$.

Authors' addresses: Astrid Bunge, TU Dortmund University, Dortmund, Germany, astrid.bunge@tu-dortmund.de; Philipp Herholz, ETH Zurich, Zurich, Switzerland, ph.herholz@gmail.com; Olga Sorkine-Hornung, ETH Zurich, Zurich, Switzerland, sorkine@inf.ethz.ch; Mario Botsch, TU Dortmund University, Dortmund, Germany, mario.botsch@tu-dortmund.de; Michael Kazhdan, Johns Hopkins University, Baltimore, MD, USA, misha@cs.jhu.edu.

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- We set $\mathcal{B}^d \equiv \mathcal{B}\big(\Sigma(C^d)\big) = \{\psi^d_{\eta,d}\}_{\eta\in\mathcal{N}^d}$ to be the set of Lagrange basis functions defined over the simplicial refinement of the sub-complex C^d.
- We set $\mathring{\mathcal{B}}^d \subset \mathcal{B}^d$ to be the subset of Lagrange basis functions on $\Sigma(C^d)$ indexed by nodes interior to the d-dimensional cells in C^d:
$$\mathring{\mathcal{B}}^d = \{\psi^d_{\eta,d}\}_{\eta\in\mathcal{N}^d\setminus\mathcal{N}^{d-1}}.$$

The Quadratic Energy

- Given a cell complex C, given a dimension $1 \le d \le D$, and given $\varepsilon > 0$, we define a quadratic energy $\mathcal{Q}_d(\cdot)$ on the space of functions on $\Sigma(C^d)$:
$$\mathcal{Q}_d\left(\psi^d\right) = \sum_{\sigma\in\Sigma_{d-1}(C^d)\setminus\Sigma_{d-1}(C^{d-1})} \int_\sigma \left\|\nabla^+_\sigma\psi^d - \nabla^-_\sigma\psi^d\right\|^2\, d\sigma + \sum_{\sigma\in\Sigma_d(C^d)} \varepsilon\cdot\int_\sigma \left\|\nabla\psi^d\right\|^2\, d\sigma$$
The first integral measures the C^1-continuity of ψ^d, and is taken over all $(d-1)$-dimensional simplices in the refinement of C^d that are not in the refinement of C^{d-1}. The second integral is a Dirichlet regularizer and is taken over all d-dimensional simplices in the refinement of C^d.
- We denote by $\langle\cdot,\cdot\rangle_d$ the symmetric, positive, semi-definite form defined by $\mathcal{Q}_d$ and write the energy as $\|\cdot\|^2_d \equiv \mathcal{Q}_d(\cdot)$.

Given the quadratic energy, we denote by $\mathcal{B}^d_{d-1} \subset \text{Span}(\mathcal{B}^d)$ the "coarse" basis on $\Sigma(C^d)$. This is the set of functions $\mathcal{B}^d_{d-1} = \{\psi^d_{\eta,d-1}\}$, defined on $\Sigma(C^d)$ but indexed by nodes $\eta \in \mathcal{N}^{d-1}$, where each $\psi^d_{\eta,d-1}$ minimizes the energy, subject to the interpolation constraint:
$$\begin{aligned}\psi^d_{\eta,d-1} &= \arg\min_{\psi^d\in\text{Span}(\mathcal{B}^d)} \left\|\psi^d\right\|^2_d\\ \text{s.t.}\quad & \psi^d_{\eta,d-1}(\eta') = \delta_{\eta\eta'},\quad \forall\eta'\in\mathcal{N}^{d-1}.\end{aligned}$$

2 PROPERTIES OF THE BASIS $\mathcal{B}^d_{d-1}$

In what follows, we demonstrate that the functions in $\mathcal{B}^d_{d-1}$ are unique, satisfy the partition of unity property, and (under appropriate conditions) have linear precision.

In doing so, we make use of the following lemmas.

Lemma 2.1. *The basis $\mathring{\mathcal{B}}^d$ spans the subspace of functions in Span($\mathcal{B}^d$) that vanish on $\Sigma(C^{d-1})$.*

Proof. Suppose we are given a function $\psi^d \in \text{Span}(\mathcal{B}^d)$:
$$\psi^d = \sum_{\eta\in\mathcal{N}^d} \alpha_\eta\cdot\psi^d_{\eta,d}.$$
If ψ^d vanishes on $\Sigma(C^{d-1})$ then, in particular, it must vanish at every node $\eta' \in \mathcal{N}^{d-1}$. But because the $\mathcal{B}^d$ satisfy the Lagrange interpolation property, we have $\psi^d(\eta') = \alpha_{\eta'}$. In particular, this implies that if ψ^d vanishes on $\Sigma(C^{d-1})$ then $\psi^d \in \mathring{\mathcal{B}}^d$.

Conversely, as $\mathcal{N}^{d-1} \subset \mathcal{N}^d$, given $\psi^d_{\eta,d}$ with $\eta \in \mathcal{N}^d\setminus\mathcal{N}^{d-1}$ we must have $\psi^d_{\eta,d}(\eta') = 0$ for all $\eta' \in \mathcal{N}^{d-1}$ since $\psi^d_{\eta,d}$ satisfies the interpolation property. On the other hand, since $\psi^d_{\eta,d}$ is strictly quadratic, its restriction to $\Sigma(C^{d-1})$ will also be strictly quadratic, so that the fact that it vanishes at all quadratic Lagrange nodes $\mathcal{N}^{d-1}$ implies that it must be constantly zero on $\Sigma(C^{d-1})$. □

Lemma 2.2. *Though the symmetric bilinear form $\langle\cdot,\cdot\rangle_d$ is only <u>semi</u>-definite on Span($\mathcal{B}^d$) it is strictly definite on Span($\mathring{\mathcal{B}}^d$).*

Proof. Suppose that we have $\mathring{\psi}^d \in \text{Span}(\mathring{\mathcal{B}}^d)$ such that $\|\mathring{\psi}^d\|^2_d = 0$. Since $\mathring{\psi}^d \in \text{Span}(\mathcal{B}^d)$ it must be continuous. Let $c \in C^d\setminus C^{d-1}$ be a d-dimensional cell in C. Since $\|\mathring{\psi}^d\|^2_d = 0$, it must be constant on c. And since $\mathring{\psi}^d \in \text{Span}(\mathring{\mathcal{B}}^d)$ it vanishes on the boundary $\partial\Sigma(c) \subset \Sigma(C^{d-1})$. Thus it must be the case that $\mathring{\psi}^d = 0$. □

Lemma 2.3. *The spaces Span($\mathcal{B}^d_{d-1}$) and Span($\mathring{\mathcal{B}}^d$) are orthogonal with respect to $\langle\cdot,\cdot\rangle_d$.*

Proof. It suffices to show that $\langle\psi^d_{\eta,d-1},\mathring{\psi}^d\rangle_d = 0$ for all $\eta \in \mathcal{N}^{d-1}$ and all $\mathring{\psi}^d \in \text{Span}(\mathring{\mathcal{B}}^d)$.

Suppose to the contrary that there exists $\eta \in \mathcal{N}^{d-1}$ and $\mathring{\psi}^d \in \text{Span}(\mathring{\mathcal{B}}^d)$ with $\langle\psi^d_{\eta,d-1},\mathring{\psi}^d\rangle_d \neq 0$. Consider the function
$$\psi^{d,\alpha}_{\eta,d-1} = \psi^d_{\eta,d-1} + \alpha\cdot\mathring{\psi}^d.$$
On the one hand we have $\psi^{d,\alpha}_{\eta,d-1}(\eta') = \psi^d_{\eta,d-1}(\eta') = \delta_{\eta\eta'}$ for all $\eta' \in \mathcal{N}^{d-1}$. On the other, the energy of $\psi^{d,\alpha}_{\eta,d-1}$ can be expressed as:
$$\begin{aligned}\left\|\psi^{d,\alpha}_{\eta,d-1}\right\|^2_d &= \left\|\psi^d_{\eta,d-1} + \alpha\cdot\mathring{\psi}^d\right\|^2_d\\ &= \left\|\psi^d_{\eta,d-1}\right\|^2_d + \underbrace{2\alpha\left\langle\psi^d_{\eta,d-1},\mathring{\psi}^d\right\rangle_d + \alpha^2\left\|\mathring{\psi}^d\right\|^2_d}_{P(\alpha)}.\end{aligned}$$
At $\alpha = 0$, the polynomial $P(\alpha)$ vanishes but its derivative does not. Thus, there exists a small value α^* (less than zero if $P'(0) > 0$ and greater than zero if $P'(0) < 0$) such that $P(\alpha^*)$ is negative. Setting $\tilde{\psi}^d_{\eta,d-1} = \psi^d_{\eta,d-1} + \alpha^*\cdot\mathring{\psi}^d$ we get a function with $\tilde{\psi}^d_{\eta,d-1}(\eta') = \delta_{\eta\eta'}$ for all $\eta' \in \mathcal{N}^{d-1}$, such that:
$$\left\|\tilde{\psi}^d_{\eta,d-1}\right\|^2_d < \left\|\psi^d_{\eta,d-1}\right\|^2_d,$$
contradicting the fact that $\psi^d_{\eta,d-1}$ is the interpolation-constrained minimizer of the energy. □

Claim 1 (Well-Defined). *For a given $\eta \in \mathcal{N}^{d-1}$, the function $\psi^d_{\eta,d-1}$ that satisfies the interpolation conditions $\psi^d_{\eta,d-1}(\eta') = \delta_{\eta\eta'}$ for all $\eta' \in \mathcal{N}^{d-1}$ and minimizes the energy $\|\cdot\|^2_d$ is unique.*

Proof. Suppose that there are two functions $\psi^d_{\eta,d-1}$ and $\tilde{\psi}^d_{\eta,d-1}$ that satisfy the interpolation condition and minimize the energy, with $\|\psi^d_{\eta,d-1}\|^2_d = \|\tilde{\psi}^d_{\eta,d-1}\|^2_d$. We would like to invoke Lemma 2.3 using $\psi^d = \psi^d_{\eta,d-1} - \tilde{\psi}^d_{\eta,d-1}$. As both functions satisfy $\psi^d_{\eta,d-1}(\eta') = \tilde{\psi}^d_{\eta,d-1}(\eta') = \delta_{\eta\eta'}$ for all $\eta' \in \mathcal{N}^{d-1}$, their difference is equal to zero at all nodes in $\mathcal{N}^{d-1}$ and so by Lemma 2.1 we have $\psi^d \in \text{Span}(\mathring{\mathcal{B}}^d)$.

Thus, if it were the case that $\langle \psi^d_{\eta,d-1}, \psi^d \rangle_d \neq 0$, we would have a contradiction to the optimality of $\psi^d_{\eta,d-1}$.

Suppose that $0 = \langle \psi^d_{\eta,d-1}, \psi^d \rangle_d = \langle \psi^d_{\eta,d-1}, \psi^d_{\eta,d-1} - \tilde{\psi}^d_{\eta,d-1} \rangle_d$. Then we have:

$$\left\| \psi^d_{\eta,d-1} \right\|^2_d = \left\langle \psi^d_{\eta,d-1}, \tilde{\psi}^d_{\eta,d-1} \right\rangle_d .$$

But under the assumption that $\psi^d_{\eta,d-1}$ and $\tilde{\psi}^d_{\eta,d-1}$ are both minimizers, we also have $\|\psi^d_{\eta,d-1}\|^2_d = \|\tilde{\psi}^d_{\eta,d-1}\|^2_d$ so that:

$$\begin{aligned} \left\| \psi^d \right\|^2_d &= \left\| \psi^d_{\eta,d-1} - \tilde{\psi}^d_{\eta,d-1} \right\|^2_d \\ &= \left\| \psi^d_{\eta,d-1} \right\|^2_d + \left\| \tilde{\psi}^d_{\eta,d-1} \right\|^2_d - 2 \left\langle \psi^d_{\eta,d-1}, \tilde{\psi}^d_{\eta,d-1} \right\rangle_d \\ &= 0. \end{aligned}$$

Thus, by Lemma 2.2, $\psi^d = 0$. Or, equivalently, that $\psi^d_{\eta,d-1} = \tilde{\psi}^d_{\eta,d-1}$. □

Claim 2 (Partition of Unity). *The functions $\mathcal{B}^d_{d-1}$ satisfy the partition of unity property on $\Sigma(C^d)$. In particular, setting:*

$$\psi^d_{1,d-1} \equiv \sum_{\eta \in \mathcal{N}^{d-1}} \psi^d_{\eta,d-1}$$

we have $\psi^d_{1,d-1} = 1$.

Proof. Recall that the Lagrange basis $\mathcal{B}^d$ satisfies the partition of unity property, so that setting

$$\psi^d_{1,d} \equiv \sum_{\eta \in \mathcal{N}^d} \psi^d_{\eta,d}$$

we have $\psi^d_{1,d} = 1$.

We would like to apply Lemma 2.3 with $\psi^d = \psi^d_{1,d-1} - \psi^d_{1,d}$. As above, $\psi^d \in \text{Span}(\mathring{\mathcal{B}}^d)$ so that if $\langle \psi^d_{\eta,d-1}, \psi^d \rangle_d \neq 0$ for some $\eta \in \mathcal{N}^{d-1}$ we would have a contradiction to the optimality of $\psi^d_{\eta,d-1}$.

Suppose, to the contrary, that $\langle \psi^d_{\eta,d-1}, \psi^d \rangle_d = 0$ for all $\eta \in \mathcal{N}^{d-1}$. Summing, we get:

$$0 = \sum_{\eta \in \mathcal{N}^{d-1}} \left\langle \psi^d_{\eta,d-1}, \psi^d \right\rangle_d = \left\langle \psi^d_{1,d-1}, \psi^d \right\rangle_d = \left\| \psi^d \right\|^2_d ,$$

where the last equality follows from the fact that the $\|\psi^d_{1,d}\|^2_d = 0$. Then using Lemma 2.2 it follows that $\psi^d = 0$ or, equivalently, that $\psi^d_{1,d-1} = \psi^d_{1,d} = 1$. □

Claim 3 (Linear Precision). *If the cell complex C (together with the virtual vertices) is embedded in Euclidean space, $\Sigma_0(C) \subset \mathbb{R}^n$ and if for each d-dimensional cell $c \in C^d \backslash C^{d-1}$ the vertices $\Sigma_0(c)$ all lie within a d-dimensional plane, then the basis $\mathcal{B}^d_{d-1}$ has linear precision.*

Proof. We start by observing that, since the functions in $\mathcal{B}^d_{d-1}$ satisfy the interpolation condition $\psi^d_{\eta,d-1}(\eta') = \delta_{\eta\eta'}$ for all $\eta, \eta' \in \mathcal{N}^{d-1}$, the basis has linear precision if and only if for all linear functions $L : \mathbb{R}^n \to \mathbb{R}$, we have:

$$L\big|_{\Sigma(C^d)} = \psi^d_{L,d-1} \equiv \sum_{\eta \in \mathcal{N}^{d-1}} L(\eta) \cdot \psi^d_{\eta,d-1}.$$

Because the functions in the Lagrange basis $\mathcal{B}^d$ have linear precision, we have:

$$L\big|_{\Sigma(C^d)} = \psi^d_{L,d} \equiv \sum_{\eta \in \mathcal{N}^d} L(\eta) \cdot \psi^d_{\eta,d}.$$

We would like to invoke Lemma 2.3 using $\psi^d = \psi^d_{L,d-1} - \psi^d_{L,d}$. As above, $\psi^d \in \text{Span}(\mathring{\mathcal{B}}^d)$ so that if $\langle \psi^d_{\eta,d-1}, \psi^d \rangle_d \neq 0$ for some $\eta \in \mathcal{N}^{d-1}$ we would have a contradiction to the optimality of $\psi^d_{\eta,d-1}$.

Again, supposing to the contrary that $\langle \psi^d_{\eta,d-1}, \psi^d \rangle_d = 0$ for all $\eta \in \mathcal{N}^{d-1}$, we take the weighted sum:

$$\begin{aligned} 0 &= \sum_{\eta \in \mathcal{N}^{d-1}} L(\eta) \cdot \left\langle \psi^d_{\eta,d-1}, \psi^d \right\rangle_d \\ &= \sum_{\eta \in \mathcal{N}^{d-1}} L(\eta) \cdot \left\langle \psi^d_{\eta,d-1}, \psi^d_{L,d-1} - \psi^d_{L,d} \right\rangle_d \\ &= \left\langle \psi^d_{L,d-1}, \psi^d_{L,d-1} - \psi^d_{L,d} \right\rangle_d . \end{aligned}$$

Or, equivalently:

$$\left\| \psi^d_{L,d-1} \right\|^2_d = \left\langle \psi^d_{L,d-1}, \psi^d_{L,d} \right\rangle_d .$$

Using the Cauchy-Schwarz inequality it follows that:

$$\left\| \psi^d_{L,d-1} \right\|^2_d \leq \left\| \psi^d_{L,d} \right\|^2_d .$$

On the other hand, since each d-dimensional cell $c \in C^d \backslash C^{d-1}$ lies within a d-dimensional plane, we know that the restriction of the function L to c has continuous gradient. Thus, we have:

$$\left\| \psi^d_{L,d} \right\|^2_d = \sum_{\sigma \in \Sigma_d(C^d)} \varepsilon \cdot \int_\sigma \left\| \nabla \psi^d_{L,d} \right\|^2 \, d\sigma.$$

Furthermore, since $\psi^d_{L,d}$ is linear, it is harmonic within each cell $c \in C^d \backslash C^{d-1}$. Thus, of all continuous functions that agree with L on $\Sigma(C^{d-1})$, the function $\psi^d_{L,d}$ is the one minimizing the Dirichlet energy. In particular, since $\psi^d_{L,d-1}$ and $\psi^d_{L,d}$ agree with L on $\Sigma(C^{d-1})$, we have:

$$\begin{aligned} \left\| \psi^d_{L,d} \right\|^2_d &= \sum_{\sigma \in \Sigma_d(C^d)} \varepsilon \cdot \int_\sigma \left\| \nabla \psi^d_{L,d} \right\|^2 \\ &\leq \sum_{\sigma \in \Sigma_d(C^d)} \varepsilon \cdot \int_\sigma \left\| \nabla \psi^d_{L,d-1} \right\|^2 \leq \left\| \psi^d_{L,d-1} \right\|^2_d . \end{aligned}$$

Combining the two inequalities, we get:

$$\left\| \psi^d_{L,d-1} \right\|^2_d = \left\| \psi^d_{L,d} \right\|^2_d = \left\langle \psi^d_{L,d-1}, \psi^d_{L,d} \right\rangle_d .$$

As above, this implies that:

$$\left\| \psi^d_{L,d-1} - \psi^d_{L,d} \right\|^2_d = 0$$

and using Lemma 2.2 it follows that $\psi^d_{L,d-1} = \psi^d_{L,d} = L\big|_{\Sigma(C^d)}$. □

3 GENERAL PROLONGATION

The basis $\mathcal{B}^d_{d-1}$ consists of functions indexed by nodes $\eta \in \mathcal{N}^{d-1}$ that are defined on $\Sigma(C^d)$. We now describe an approach for generalizing this formulation, defining a basis $\mathcal{B}^d_{d'}$ for all $0 \le d' < d \le D$, where the basis functions are indexed by nodes $\eta' \in \mathcal{N}^{d'}$ and defined on $\Sigma(C^d)$. As above, we show that these bases can be expressed as the solution to a constrained minimization problem, satisfy the partition of unity property, and (under appropriate conditions) satisfy the partition of unity property.

Definition 3.1. Given the basis $\mathcal{B}^d_{d-1}$, we define the prolongation matrix $\mathbf{P}^d_{d-1} \in \mathbb{R}^{|\mathcal{N}^d|\times|\mathcal{N}^{d-1}|}$ to be the matrix whose coefficients give the expression of $\psi^d_{\eta',d-1}$ as a linear combination of the $\psi^d_{\eta,d}$:

$$\psi^d_{\eta',d-1} = \sum_{\eta\in\mathcal{N}^d} \left(\mathbf{P}^d_{d-1}\right)_{\eta\eta'} \psi^d_{\eta,d}.$$

Definition 3.2. For $0 \le d' < d \le D$, we define the prolongation matrix $\mathbf{P}^d_{d'}$ to be the composition:

$$\mathbf{P}^d_{d'} = \mathbf{P}^d_{d-1} \cdots \mathbf{P}^{d'+1}_{d'}.$$

Definition 3.3. For $0 \le d' < d \le D$ we define $\mathcal{B}^d_{d'} = \{\psi^d_{\eta',d'}\}_{\eta'\in\mathcal{N}^{d'}}$ to be the subset of Span($\mathcal{B}^d$) such that:

$$\psi^d_{\eta',d'} \equiv \sum_{\eta\in\mathcal{N}^d} \left(\mathbf{P}^d_{d'}\right)_{\eta\eta'} \cdot \psi^d_{\eta,d}.$$

We note that for all d' with $0 \le d' < d$ we have $\mathcal{B}^d_{d'} \subset \text{Span}(\mathcal{B}^d_{d-1})$. This motivates the following claim.

Claim 4 (Optimality). *Given $0 \le d' < d \le D$ and given $\eta' \in \mathcal{N}^{d'}$, if $\tilde\psi^d_{\eta',d'} \in$ Span($\mathcal{B}^d$) is a function that agrees with $\psi^d_{\eta',d'}$ on $\Sigma(C^{d-1})$ then we have:*

$$\left\|\psi^d_{\eta',d'}\right\|^2_d \le \left\|\tilde\psi^d_{\eta',d'}\right\|^2_d,$$

with equality if and only if $\psi^d_{\eta',d'} = \tilde\psi^d_{\eta',d'}$. Thus, of all functions in Span($\mathcal{B}^d$) that agree with $\psi^d_{\eta',d'}$ on $\Sigma(C^{d-1})$ the function $\psi^d_{\eta',d'}$ is the unique minimizer of the energy $\|\cdot\|^2_d$.

Proof. Consider the difference $\psi^d = \tilde\psi^d_{\eta',d'} - \psi^d_{\eta',d'}$. As $\tilde\psi^d_{\eta',d'}$ and $\psi^d_{\eta',d'}$ agree on $\Sigma(C^{d-1})$ we have $\psi^d \in \text{Span}(\mathring{\mathcal{B}}^d)$.

Expanding the energy of $\tilde\psi^d_{\eta',d'}$ we get:

$$\begin{aligned}\left\|\tilde\psi^d_{\eta',d'}\right\|^2_d &= \left\|\psi^d_{\eta',d'} + \psi^d\right\|^2_d \\ &= \left\|\psi^d_{\eta',d'}\right\|^2_d + \left\|\psi^d\right\|^2_d + 2\left\langle\psi^d_{\eta',d'},\psi^d\right\rangle_d \\ &= \left\|\psi^d_{\eta',d'}\right\|^2_d + \left\|\psi^d\right\|^2_d\end{aligned}$$

where the last equality follows from Lemma 2.3 – using the fact that $\psi^d_{\eta',d'} \in \text{Span}(\mathcal{B}^d_{d-1})$ and $\psi^d \in \text{Span}(\mathring{\mathcal{B}}^d)$. Thus the energy of $\psi^d_{\eta',d'}$ is no greater than the energy of $\tilde\psi^d_{\eta',d'}$, with equality if and only if the energy of ψ^d is zero. But since $\psi^d \in \text{Span}(\mathring{\mathcal{B}}^d)$, Lemma 2.2 implies that the energy of ψ^d vanishes if and only if $\psi^d = 0$ or, equivalently, if and only if $\psi^d_{\eta',d'} = \tilde\psi^d_{\eta',d'}$. □

Claim 5 (Partition of Unity). *If the basis $\mathcal{B}^{d-1}_{d'}$ satisfies the partition of unity property, then so does the basis $\mathcal{B}^d_{d'}$.*

Proof. Consider the functions:

$$\psi^d_{1,d'} \equiv \sum_{\eta'\in\mathcal{N}^{d'}} \psi^d_{\eta',d'} \in \mathcal{B}^d \qquad \text{and} \qquad \psi^d_{1,d} \equiv \sum_{\eta\in\mathcal{N}^d} \psi^d_{\eta,d}.$$

We have $\psi^d_{1,d'} \in \text{Span}(\mathcal{B}^d_{d-1})$ and, by the assumption of the claim, the functions $\psi^d_{1,d'}$ and $\psi^d_{1,d}$ are both constantly equal to one on $\Sigma(C^{d-1})$. By Claim 4 we have:

$$\left\|\psi^d_{1,d'}\right\|^2_d \le \left\|\psi^d_{1,d}\right\|^2_d = 0.$$

Thus, we must have $\|\psi^d_{1,d'}\|^2_d = \|\psi^d_{1,d}\|^2_d$ so, by the claim, the two functions are equal – $\psi^d_{1,d'} = \psi^d_{1,d} = 1$. □

Corollary 3.3.1. *Using the fact that $\mathcal{B}^{d'+1}_{d'}$ satisfies the partition of unity property, it follows that $\mathcal{B}^d_{d'}$ satisfies the partition of unity property.*

Claim 6 (Linear Precision). *If the basis $\mathcal{B}^{d-1}_{d'}$ satisfies the linear precision property and every d-dimensional cell $c \in C^d \backslash C^{d-1}$ has the property that the vertices of its simplicial refinement $\Sigma_0(c)$ lie in a d-dimensional plane, then the basis $\mathcal{B}^d_{d'}$ also has linear precision.*

Proof. Let $L : \mathbb{R}^n \to \mathbb{R}$ be a linear function and consider the functions $\psi^d_{L,d'}$ and $\psi^d_{L,d}$:

$$\begin{aligned}\psi^d_{L,d'} &\equiv \sum_{\eta'\in\mathcal{N}^{d'}} L(\eta')\cdot\psi^d_{\eta',d'} \in \mathcal{B}^d_{d'} \\ \psi^d_{L,d} &\equiv \sum_{\eta\in\mathcal{N}^d} L(\eta)\cdot\psi^d_{\eta,d} = L\big|_{\Sigma(C^d)}.\end{aligned}$$

By the assumption of the claim we know that $\psi^d_{L,d'}$ agrees with $\psi^d_{L,d}$ on the simplicial complex $\Sigma(C^{d-1})$. Thus, by Claim 4 we know that:

$$\left\|\psi^d_{L,d'}\right\|^2_d \le \left\|\psi^d_{L,d}\right\|^2_d.$$

On the other hand, we know that of all functions agreeing with L on $\Sigma(C^{d-1})$, the function $\psi^d_{L,d}$ minimizes $\|\cdot\|^2_d$ since it is C^1 and harmonic. Thus the two functions must have the same energy, $\|\psi^d_{L,d'}\|^2_d = \|\psi^d_{L,d}\|^2_d$ so, by the claim, the functions are equal – $\psi^d_{L,d'} = \psi^d_{L,d} = L\big|_{\Sigma(C^d)}$. □

Corollary 3.3.2. *Given $1 \le d \le D$, if for every $1 \le d' \le d$, every d'-th dimensional cell $c \in C^{d'} \backslash C^{d'-1}$ has the property that the vertices of the simplicial refinement $\Sigma_0(c)$ lie in a d'-dimensional plane, then the basis $\mathcal{B}^d_{d'}$ has linear precision.*

4 DISCUSSION

The Need for gradient continuity

In these derivations we only used the Dirichlet regularizer in the definition of the energies $Q_d(\cdot)$ and could have foregone the gradient-continuity term, as all the functions we ended up considering had continuous gradients in any case.

The reason we incorporate the gradient continuity energy is to prove that the basis $\mathcal{B}^d_{d-1}$ reproduces the Lagrange basis in the case that C is a simplicial complex. That is, that our work generalizes the standard Lagrange basis. In that case the reproduction proof holds when the energy is defined entirely in terms of gradient continuity (i.e. when $\varepsilon = 0$). Thus, in practice our works assumes that ε is an arbitrarily small, but non-zero, value. It needs to be non-zero for the proofs described here to hold. It needs to be arbitrary small so that we come arbitrarily close to reproducing the Lagrange basis in the case that the cell complex is a simplicial complex. (One could make the reproduction exact by identifying the kernel of the gradient continuity energy and only adding the Dirichlet energy of the projection onto the kernel. However, this would require identifying the kernel for each cell, making the implementation slower in practice.)

The Order of the Shape Functions

Though our research focuses specifically on quadratic Lagrange elements, the proofs above only assume that the order is at least one, so that we are working within a continuous function space.

The Choice of Metric

The definition of Q_d requires the choice of a metric in order to define gradients and compute integrals. In general, our implementation only requires an assignment of a symmetric positive definite matrix with each D-dimensional simplex of the refinement, $\sigma \in \Sigma_D(C)$.

However, in the case that the cell complex C is embedded in $\mathbb{R}^n$ and we require linear precision, we assume that the metric is induced by the embedding. This ensures that the restriction of linear functions to the cell complex will have continuous gradients.

We note that the piecewise-constant assignment of metric tensors to simplices does not ensure that the induced metric on boundary simplices is consistently defined for face-adjacent simplices. This *is* the case when the metric is induced by the embedding of cell complex in $\mathbb{R}^n$ but need not be true in general. In our application of anisotropic diffusion (for line integral convolution) this is not an issue, but it could be in other contexts.[1] One can bypass this problem by defining the metric by assigning lengths to all edges of the simplicial complex. This ensures metrics are consistently defined on boundary faces, but has the limitation that the assignment of positive edge weights must satisfy the triangle inequality – a nonlinear constraint on the set of edge weights.

Planarity Testing

As discussed above, if the cell complex is embedded in Euclidean space, $\Sigma_0(C) \subset \mathbb{R}^N$, and we would like to have linear precision, we only need to explicitly impose linear precision constraints for those d-dimensional cells $c \in C$ whose vertices do not lie within a d-dimensional plane. This can be checked by constructing the characteristic polynomial of the covariance matrix of the vertices of c and checking that the lowest $N - d$ coefficients are zero. That is that is, that the characteristic polynomial $P(x)$ is divisible by x^{N-d}.

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[1] Our approach supports this metric discontinuity, with the implication being that gradient discontinuity can be manifest along the virtual face, not just across it.